SMOOTH MANIFOLDS FALL 2023 - HOMEWORK 4

SOLUTIONS

Problem 1. Let X be a smooth manifold and $f : X \to \mathbb{R}$ by C^∞. Show that T^*X is a vector bundle over X. Furthermore, show that the family of functions defined by $\theta_x(v) = D_v(f)$ is a C^∞ section of T^*M, where $v \in T_xM$ and D_v is the derivation determined by v. θ_x is called the *differential of* f and is often written df.

Solution. We build charts for T^*X explicitly. Let $\varphi : U \to \mathbb{R}^n$ be a chart of X. Let $\hat{U} \subset T^*X$ be the set of linear functionals on T_xX for some $x \in U$, and define $\hat{\varphi} : \hat{U} \to \mathbb{R}^n \times \mathbb{R}^n$ by

$$\hat{\varphi}(\theta) = (\varphi(x), \theta(d\varphi^{-1}(\varphi(x))(e_1)), \ldots, \theta(d\varphi^{-1}(\varphi(x))(e_n)).$$

Notice that $\hat{\varphi}(\theta_1) = \hat{\varphi}(\theta_2)$ if and only if θ_1 and θ_2 are defined on the same vector space $T_{\theta_i(x)}$, and θ_1 and θ_2 agree on a basis (since $d\varphi^{-1}(\varphi(x))$ is an isomorphism at every x). Hence, $\theta_1 = \theta_2$ and $\hat{\varphi}$ is bijective. It is also clear that $\hat{\varphi}$ is linear when restricted to a single vector space T_y^*M, $y \in U$.

We show that transition maps $\hat{\varphi} \circ \hat{\psi}^{-1}$ are C^∞. Indeed, we first note that if $A_x = d\psi(x) \circ d\varphi^{-1}(\varphi(x))$, $\theta \circ d\varphi^{-1}(\varphi(x)) = \theta \circ d\psi^{-1}(\psi(x))A_x$.

$$\hat{\varphi} \circ \hat{\psi}^{-1}(x, v) = (\varphi \circ \psi^{-1}(x), A_x v).$$

Indeed, if $\theta(d\psi^{-1}(\psi(x))e_i) = v_i$, and $w_i = \theta(d\varphi^{-1}(\varphi(x))e_i) = \theta(d\psi^{-1}(\psi(x))A_x e_i)$, then

$$w_i = \theta\left(d\psi^{-1}(\psi(x)) \sum_{j=1}^{n} (A_x)_{ji} e_i\right) = \sum_{j=1}^{n} (A_x)_{ji} v_i$$

It is now clear that $\hat{\varphi} \circ \hat{\psi}^{-1}$ is C^∞.

Finally, let $f : X \to \mathbb{R}$ be C^∞, and $\theta_x(v) = D_v(f)$ for $v \in T_xM$ as described. First, note that this is a section of T^*X, since each θ_x is a linear function on T_xM for every $x \in M$. To see smoothness, we check with charts. Let φ be a chart of X and $\hat{\varphi}$ be the corresponding chart of T^*X. Then we must show that $\hat{\varphi} \circ \theta \circ \varphi^{-1}$ is C^∞ from U to $\hat{U}$, where U is the range of φ (where $\theta(x) = \theta(x, v)$. Indeed, we claim that

$$\hat{\varphi} \circ \theta \circ \varphi^{-1}(x) = (x, \nabla(f \circ \varphi^{-1})(x)) \tag{0.1}$$

where ∇f is the usual gradient on $\mathbb{R}^n$. Indeed, observe that

$$\hat{\varphi} \circ \theta(y) = (\varphi(y), D_{d\varphi^{-1}(e_1))}(f)), ..., D_{d\varphi^{-1}(e_n))}(f))).$$

Notice also that by the chain rule, $D_{d\varphi^{-1}(e_i)}(f) = \dfrac{\partial(f \circ \varphi^{-1})}{\partial i}$, so the last n components are exactly $\nabla(f \circ \varphi^{-1})(y)$. Finally, we must evaluate this function at $\varphi^{-1}(x)$, yielding (0.1). Since all partial derviatives of a C^∞ function are again C^∞, it follows that the section is C^∞. □

Problem 2. Show that if $g : \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $g(z) \neq 0$ for any $z \in \mathbb{R}$, then the integral curves of the vector field $V = xg(xy)\frac{\partial}{\partial x} - yg(xy)\frac{\partial}{\partial y}$ on $\mathbb{R}^2 \setminus \{0\}$ are the level sets of the function $F(x, y) = xy$. [*Hint*: Show that the level sets of F and integral curves of V have the same tangent bundles]

Solution. Fix $p \in \mathbb{R}^2$ and let $\gamma_p(t)$ be the integral curve of x under V. Then consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) = F(\gamma_p(t))$. It follows that

$$f'(t) = dF(\gamma_p'(t)) = dF(V(\gamma_p(t)) = \begin{pmatrix} y & x \end{pmatrix} \begin{pmatrix} x_t g_t \\ -y_t g_t \end{pmatrix} = x_t y_t g_t - y_t x_t g_t = 0$$

where $\gamma_p(t) = (x_t, y_t)$. It follows that $\gamma_p(t)$ is an immersed 1-manifold contained in $F^{-1}(F(p))$, which is a 1-manifold by the submersion theorem. Thus, it must be equal to the connected component of $F^{-1}(F(p))$ containing p, as the curves $\gamma_p(t)$ are exactly charts of the manifold! □

Problem 3. Let $F : (-\varepsilon, \varepsilon) \times X \to X$ be any C^∞ map such that $F(t, F(s, x)) = F(t + s, x)$ whenever $|t|, |s|$ and $|t + s|$ are all less than ε.

(i) Show that $V(x) := \frac{\partial}{\partial t}|_{t=0}F(t, x)$ is a C^∞ vector field on X.
(ii) Show that F is the flow generated by V.
(iii) Show that F extends uniquely to a globally defined flow $F : \mathbb{R} \times M \to M$ satisfying the flow equation [*Hint*: If $k \in \mathbb{Z}$ and $\delta \in (0, \varepsilon)$, define $F(k\varepsilon/2 + \delta, x) = T^k(F(\delta, x))$, where $T(x) = F(\varepsilon/2, x)$].
(iv) Which assumption(s) is/are not satisfied for flows which reach the boundary in finite time?

Solution.

(i) To see that V is a C^∞ vector field, we work in charts. Let $\varphi : U \to \mathbb{R}^n$ be a chart of X, so that $\bar{\varphi} : (-\varepsilon, \varepsilon) \times U \to \mathbb{R} \times \mathbb{R}^n$ defined by $\bar{\varphi}(t, x) = (t, \varphi(x))$ is a chart of $(-\varepsilon, \varepsilon) \times X$. Then $V(x) = d\varphi^{-1}\left(\dfrac{\partial F}{\partial t}F(x, 0)\right)$. Since F is C^∞, the partial derivative is a C^∞ function taking values in $\mathbb{R}^n$, which we identify with $T_{\varphi(x)}\mathbb{R}^n$. Hence, V is C^∞.
(ii) That F is the flow generated by V follows from the definition.
(iii) Observe that k is a discrete parameter and δ is a continuous parameter in the definition of the global extension of F. For every $t \in \mathbb{R}$, there exists a value of k and δ_0 such that $k\varepsilon/2 + \delta_0 = t$, and $\delta \in (0, \varepsilon)$ can vary both positively and negatively. Furthermore, for a fixed k, F is a C^∞ in x and δ since it is the composition of the time $\varepsilon/2$-map T (which is C^∞) and the flow $F(\delta, x)$, and for every t. Thus, it is C^∞ once it is well-defined.

To see that it is well-defined, note that there are at most two ways to represent a real number t and $k_1\varepsilon/2 + \delta_1$ and $k_2\varepsilon/2 + \delta_2$, since in this case, $(k_1 - k_2)\varepsilon/2 = \delta_1 - \delta_2$. Since $-\varepsilon < \delta_1 - \delta_2 < \varepsilon$, it follows that $|k_1 - k_2| \leq 1$, as the inequalities are strict. Assume without loss of generality that $k_2 = k_1 + 1$, so that $\delta_2 = \delta_1 - \varepsilon/2$. Then

$$T^{k_2}(F(\delta_2, x)) = T^{k_1+1}(F(\delta_1 - \varepsilon/2, x)) = T^{k_1}TF(-\varepsilon/2, F(\delta_1, x)) = T^{k_1}TT^{-1}F(\delta_1, x)) = T^{k_1}F(\delta_1, x).$$

Thus, F is well-defined.

□

Problem 4. Show that if X is a compact manifold and V is a smooth vector field on M, then there exists a globally defined flow φ_t^V [*Hint*: Fix a finite open cover of charts of X, and let $T(x)$ be the largest $\varepsilon > 0$ such that the flow is defined on $(-\varepsilon, \varepsilon)$ in *some* chart from the finite cover. Show that T is continuous and positive, hence bounded below. Apply the previous problem.]

Solution. Let $\mathcal{U}$ be an open cover of X by domains of charts. For each $\varepsilon > 0$, let $V_\varepsilon \subset X$ denote the set of points for which the flow is defined for time at least ε in a chart from the open cover $\mathcal{U}$. $\{V_\varepsilon : \varepsilon > 0\}$ is another open cover of X. Since it is nested in ε, there exists ε_0 such that $V_{\varepsilon_0} = X$. Thus, we have verified the assumptions of Problem 3, as there is a uniform time for which the flow is defined for all $x\ inX$. □